1. (a) Suppose that T is a projection.

Then ~~?~~ $v \in \operatorname{im} T$ iff $Tv = v$.

For, if $v \in \operatorname{im} T$, then $v = Tw$ for some $w \in V$,

so then $Tv = T^2 w = Tw = v$

since $T^2 = T$.

The reverse implication is trivial.

Now $\ker T \cap \operatorname{im} T = \{0\}$,

for if $v \in \operatorname{im} T$, then $Tv = v$ as above,

and if $v \in \ker T$, then $Tv = 0$,

so $v = 0$.

Finally if $v \in V$, then $v = Tv + v - Tv$;

$Tv \in \operatorname{im} T$ and $v - Tv \in \ker T$

since $T(v - Tv) = Tv - T^2 v$

$= Tv - Tv$

$= 0$.

So $V = \ker T \oplus \operatorname{im} T$ as required. [3 marks]

Now every element of $\ker T$ is an eigenvector with eigenvalue 0, and we have already seen that every element of $\operatorname{im} T$ is an eigenvector with eigenvalue 1. So there is a spanning set of eigenvectors & therefore a basis of eigenvectors. So T is diagonalisable.

[2 marks]

The above argument shows that there are no eigenvalues other than 0 and 1; so does the observation that since $T^2=T$, $T^2-T=0$, so $m_T(x) \mid x^2-x = x(x-1)$. ~~[2 marks]~~

The characteristic polynomial is $x^k(1-x)^\ell$, where $k=\dim\ker T$ and $\ell = \dim \operatorname{im} T$. [1 mark]

The minimum polynomial is

$$\begin{cases} x & \text{if } \ell=0 \quad (\text{so } T=0) \\ 1-x & \text{if } k=0 \quad (\text{so } T=I) \\ x(1-x) & \text{otherwise.} \end{cases}$$

[2 marks]

(This is all on the problem sheets.)

$m_{\alpha I+\beta T}(x)$ is $(x-\alpha)$, if $T=0$, $(x-\alpha-\beta)$, if $T=I$, & $(x-\alpha)(x-\alpha-\beta)$ o/w. [2 marks]

(b) (i) $E+F$ is a projection iff

$$(E+F)^2 = E+F.$$

Now $(E+F)^2 = E^2+EF+FE+F^2$

$= E+EF+FE+F$

since E and F are projections,

[3 marks]

so $(E+F)^2 = E+F$

iff $EF+FE=0$, that is $EF=-FE$.

(ii). If $\mathbb{F}$ does not have characteristic 2, then/the above implies that $0 = E(EF+FE) = E^2F + EFE$

$= EF + EFE,$

while $0 = (EF+FE)E = EFE + FE^2$
$= EFE + FE,$

so that $EF = FE$.

Since $EF = -FE$,

$2EF = 0$ so $EF = 0$ since $2 \neq 0$.

Hence also $FE = 0$.

Conversely if $EF = FE = 0$,
then $EF = -FE$ [4 marks]
so by (i) $E+F$ is a projection.

(iii) In $(\mathbb{Z}_2)^3$, consider the three matrices

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

A, B and C all represent projections, and
$C = A+B$, while $AB \neq 0$. [3 marks]
(Familiar stuff, from past papers)

(c). The same three matrices, over a field not of characteristic 2, give a counterexample since A, B and C commute, $ABC = 0$,

but $A+B+C = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$ which is not the matrix of a projection since 2 is an eigenvalue.

(New) [5 marks]

2(a) T is diagonalisable iff $m_T(x)$ is a product of distinct (monic) linear factors. (1)
(Bookwork) [2 marks]

$$\chi_A(x) = \begin{vmatrix} x & -1 & 0 \\ -6 & x & 2 \\ -4 & -6 & x+3 \end{vmatrix}$$

[or similarly with all terms multiplied by -1]

$$= x \begin{vmatrix} x & 2 \\ -6 & x+3 \end{vmatrix} + \begin{vmatrix} -6 & 2 \\ -4 & x+3 \end{vmatrix}$$

$$= x\left(x(x+3)+12\right) + (-6)(x+3) + 8$$

$$= x^3 + 3x^2 + 12x - 6x - 18 + 8$$

$$= x^3 + 3x^2 + 6x - 10.$$

By inspection 1 is a root. Then

$$x^3 + 3x^2 + 6x - 10 = (x-1)(x^2+4x+10),$$

and the roots of this are 1 and

$$\frac{-2 \pm \sqrt{16-40}}{2} = -1 \pm \sqrt{-6},$$

in any field not of characteristic 2 in which -6 has a square root; if the field is not of characteristic 2 and -6 has no square root, then $x^2+4x+10$ is irreducible.

(i). A is diagonalisable over $\mathbb{C}$ because $\chi_A(x)$ is a product of distinct linear factors and so by the Cayley-Hamilton Theorem $m_A(x)$ must be also.

[Or: A has 3 distinct eigenvalues and is 3×3, so it is diagonalisable.]

[2 marks]

(ii). Over $\mathbb{R}$, $x^2+4x+10$ is irreducible. Since $A \neq I$, $m_A(x) \neq x-1$, so

$x^2+4x+10 \mid m_A(x)$, so A is not diagonalisable. [2 marks]

[Or: laboriously check that A has only a 1-dimensional eigenspace; or: use that every irreducible factor of $\chi_A(x)$ is a factor of $m_A(x)$.]

~~(iii). By the same reasoning, A is not diagonalisable over $\mathbb{Q}$~~.

(iii). Over $\mathbb{Z}_3$, $-6=0$, so $\chi_A(x) = (x-1)^3$.

Since $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \neq I$, $A - I \neq 0$

so $m_A(x) \neq x-1$. So $m_A(x)$ has a repeated linear factor so A is not diagonalisable over $\mathbb{Z}_3$.

[2 marks]

(iv) In $\mathbb{Z}_2$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

and $\chi_A(x) = x^3 + 3x^2 + 6x - 10$

$= x^2(x+1)$.

It is clear that $A(A+I) \neq 0$,

so $m_A(x) \nmid x(x+1)$,

so $m_A(x)$ has a repeated linear factor

& A is not diagonalisable.

[2 marks].

(b). Primary Decomposition Theorem:

Suppose that $m_T(x) = p(x)q(x)$, where $p(x)$ & $q(x)$ are coprime.

Then $V = \ker p(T) \oplus \ker q(T)$,

$m_{T|\ker p(T)}(x) = p(x)$, and $m_{T|\ker q(T)}(x) = q(x)$.

(Bookwork) [2 marks]

It is easy from this to prove that if $m_T(x) = p_1(x)p_2(x)\dots p_r(x)$, where the $p_i(x)$ are mutually coprime, then

$$V = \ker p_1(T) \oplus \ker p_2(T) \oplus \dots \oplus \ker p_r(T),$$

and $m_{T|\ker p_i(T)}(x) = p_i(x)$.

If T is diagonalisable, then $m_T(x)$ is a product of distinct linear ~~transformats~~ factors. Write

$$m_T(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_r).$$

Then by PDT,

$$V = \bigoplus_{i=1}^{r} \ker(T-\lambda_i I)$$, since the $x-\lambda_i$ are coprime,

and so $V = \bigoplus_{i=1}^{r} V_{\lambda_i}$.

This is equivalent to the given form since if $\lambda \notin \{\lambda_1, \dots \lambda_r\}$, then $\ker(T-\lambda I) = \{0\}$.

[3 marks]

Suppose that S & T commute, and that $v \in V_\lambda$.

Then $TSv = STv = S(\lambda v) = \lambda Sv$.

So $Sv \in V_\lambda$ as required.

Now S is diaganalisable, so $m_S(x)$ is a product of distinct linear factors; clearly $m_{S|V_\lambda}(x) \mid m_S(x)$ so $m_{S|V_\lambda}(x)$ is a product of distinct linear factors so $S|_{V_\lambda}$ is diaganalisable.

Let $\mathcal{B}_\lambda$ be a basis for ~~$\mathcal{B}$~~ V_λ consisting of eigenvectors of $S|_{V_\lambda}$.

Then if $\mathcal{B} = \bigcup_\lambda \mathcal{B}_\lambda$, then $\mathcal{B}$ is a basis for V all of whose elements are eigenvectors of both S & T. [3 marks]

Now suppose that $\mathcal{B}$ is a basis with respect to which ${}_{\mathcal{B}}[S]_{\mathcal{B}}$ and ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ are diagonal.

Then clearly these two matrices commute, so so do S and T. [2 marks]

(familiar stuff)

(c). Suppose that S_1, S_2 and S_3 are diaganalisable. For each $\lambda, \mu \in \mathbb{F}$, let

$$V_\mu = \ker(S_3 - \mu I),$$

and $V_{\lambda,\mu} = \ker(S_2 - \lambda I) \cap \ker(S_3 - \mu I)$.

Exactly as above, ~~there~~

$$V = \bigoplus_{\lambda,\mu} V_{\lambda,\mu}.$$

As above, S_1 commutes with S_2 & S_3 so ~~fixes~~ $\ker(S_2 - \lambda I)$ and $\ker(S_3 - \mu I)$ are S_1-invariant and so so is $V_{\lambda,\mu}$.

As above let $\mathcal{B}_{\lambda,\mu}$ be a basis of eigenvectors for $S_1|_{V_{\lambda,\mu}}$ for $V_{\lambda,\mu}$.

Then $\mathcal{B} = \bigcup_{\lambda,\mu} \mathcal{B}_{\lambda,\mu}$ is a basis for V consisting of vectors which are simultaneously eigenvectors of S_1, S_2 and S_3.

(New) [5 marks].

3.(a).(i). e_i' is defined so that

$$e_i'(e_j) = \delta_{ij}. \quad [2 \text{ marks}]$$

(ii). Define $\Phi: V \to V''$ so that for all $v \in V$ & $f \in V'$,

$$\Phi(v)(f) = f(v).$$

(α) $\Phi(v) \in V''$:

Suppose that $f, g \in V'$ and $\alpha, \beta \in \mathbb{C}$.

Then $\Phi(v)(\alpha f + \beta g)$

$= (\alpha f + \beta g)(v)$ by definition of Φ

$= \alpha f(v) + \beta g(v)$ by definition of the vector space operations on V'

$= \alpha \Phi(v)(f) + \beta \Phi(v)(g)$ by definition of Φ.

So $\Phi(v)$ is linear. Since $\operatorname{ran} \Phi(v) \subseteq \mathbb{C}$, $\Phi(v) \in V''$.

(β) Φ is linear:

Suppose that ~~$v \in V$,~~ $u, v \in V$, $\alpha, \beta \in \mathbb{C}$, $f \in V'$.

Then $\Phi(\alpha u + \beta v)(f) = f(\alpha u + \beta v)$ by definition of Φ

$= \alpha f(u) + \beta f(v)$ since $f \in V'$ & so is linear

$= \alpha \Phi(u)(f) + \beta \Phi(v)(f)$ by definition of Φ

$= (\alpha \Phi(u) + \beta \Phi(v))(f)$ by definition of the v. sp. ops on V''.

So the functions $\Phi(\alpha u + \beta v)$ and $\alpha \Phi(u) + \beta \Phi(u)$ are equal.

So Φ is linear.

(γ). Φ is ~~onto~~ 1-1:

Suppose $\boxed{\cancel{v \in \ker \Phi}}$ $v \neq 0$.

Say $v = \sum_{i=1}^{n} \alpha_i e_i$, and $\alpha_i \neq 0$.

Then $e_i'(v) = \alpha_i \neq 0$. (*)

Hence $\Phi(v)(e_i') \neq 0$.

Hence $\ker \Phi$ is trivial so Φ is 1-1.

[Alternative to (*) : if $v \neq 0$, extend v

+ a basis $\{v, v_2 \dots v_n\}$, form a dual basis $\{f_1, f_2 \dots f_n\}$; then $f_1(v) \neq 0$.]

(δ). Now $\phi : V \to V''$ is a 1-1 linear transformation, and $\dim V'' = \dim V' = \dim V$, so Φ is onto.

So Φ is an ~~homom~~ isomorphism.

(Bookwork) [~~3~~5 marks]

(iii). Let $u = \sum_{i=1}^{n} \alpha_i e_i$ & $v = \sum_{i=1}^{n} \beta_i e_i$.

Then $\varphi(u, v) = \pi(v)(u)$

$$= \sum_{i=1}^{n} \overline{\beta_i}\, e_i' \left(\sum_{i=1}^{n} \alpha_i e_i \right)$$

$$= \sum_{i=1}^{n} \overline{\beta_i}\, \alpha_i \qquad \phi$$

φ is positive definite:

$$\varphi(u, u) = \sum_{i=1}^{n} \overline{\alpha_i}\, \alpha_i = \sum_{i=1}^{n} |\alpha_i|^2$$

which is ≥ 0 & $= 0$ iff $\alpha_i = 0$ for all i.

φ is linear in the first term:

Obvious.

φ is complex-conjugate symmetric:

Clear.

And, $\varphi(e_i, e_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{o/w}, \end{cases}$

so φ is indeed an inner product wrt to which $\{e_1 \ldots e_n\}$ is orthonormal.

[Or, from (*) spot that (V, φ) is isomorphic to $(\mathbb{C}^n, \langle , \rangle)$, so φ is
standard inner product
the innerproduct required.]

(New?) [5 marks]

(b)(i). Suppose $v \in V \setminus \{0\}$ and $Tv = \lambda v$.

Then $\lambda \langle v, v \rangle = \langle \lambda v, v \rangle$

$= \langle Tv, v \rangle$

$= \langle v, Tv \rangle$ by self-adjointness

$= \overline{\langle Tv, v \rangle}$

$= \overline{\langle \lambda v, v \rangle}$

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(b) i. Suppose v is an eigenvector of T with eigenvalue λ.

Then $\lambda\langle v, v\rangle = \langle Tv, v\rangle$

$$= \langle v, -Tv\rangle \quad \text{since } T^* = -T$$

$$= \langle v, -\lambda v\rangle$$

$$= \overline{\langle -\lambda v, v\rangle}$$

$$= \overline{-\lambda\langle v, v\rangle}$$

$$= \overline{-\lambda}\,\overline{\langle v, v\rangle}$$

$$= \overline{-\lambda}\langle v, v\rangle.$$

Since $v \neq 0$, $\lambda = \overline{-\lambda}$ so λ is purely imaginary. [3 marks]

ii. Suppose U is T-invariant & $v \in U^\perp$.

Then for all $u \in U$, $Tu \in U$, so

$$\langle Tu, v\rangle = 0$$

so $\langle u, Tv\rangle = 0$

so $Tv \perp u$.

Hence $Tv \in U^\perp$. [2 marks]

iii. We prove this by induction on $\dim V$.

If $\dim V = 0$ or 1 it is trivial.

Now suppose $\dim V > 1$.

By the Fundamental Theorem of Algebra T has an eigenvalue, λ say, which must have an eigenvector, which we label e_1.

Now $\{e_1\}^\perp$ is T-invariant by part ii., so by the inductive hypothesis there is a basis for $\{e_1\}^\perp$ consisting of eigenvectors of $T|_{\{e_1\}^\perp}$. Refer to this as $\{e_2 \cdots e_n\}$.

Then $\{e_1 \cdots e_n\}$ is a basis of eigenvectors of T.

[~~3~~ 4 marks]

(These three parts are modifications of bookwork arguments about self-adjoint linear transformations.)

iv. Suppose that A is a real antisymmetric matrix. Then considered as an element of $M_{n\times n}(\mathbb{C})$, it is diagonalisable, so its minimum polynomial is a product of distinct linear factors.

Now if $p(x)$ is any element of $\mathbb{C}[x]$,

$$\overline{p(A)} = \bar{p}(\bar{A}) = \bar{p}(A), \text{ since } A \text{ is real.}$$

Hence ~~p~~ $\cancel{m_A(xA) + m_A}$

$\overline{m_A}(A) = 0$, so since $\overline{m_A}(x)$ is monic & of the same degree as $m_A(x)$, they must be equal.

So $m_A(x)$ has real coefficients.

If now α is any root of $m_A(x)$,

$$m_A(\alpha) = 0 \quad \text{so} \quad \overline{m_A(\alpha)} = 0 \quad \text{so} \quad m_A(\bar{\alpha}) = 0.$$

So $\bar{\alpha}$ is also a root.

Also by i. all roots come in complex conjugate pairs.

Hence $m_A(x)$ has one of the two forms

$$x(x - i\alpha_1)(x + i\alpha_1) \ldots (x - i\alpha_r)(x + i\alpha_r)$$

if 0 is a root

$$\& \quad (x - i\alpha_1)(x + i\alpha_1) \ldots (x - i\alpha_r)(x + i\alpha_r)$$

if not.

[6 marks]

(New, but should be easy to see.)

v. $\frac{1}{\alpha} T v = \frac{1}{\alpha} T\left(\frac{1}{\alpha} T u\right) = \frac{1}{\alpha^2} T^2 u$

$= \frac{1}{\alpha^2}(-\alpha^2 u)$ since $u \in \ker(T^2 + \alpha^2 I^2)$

$= -u$,

$\langle u, v \rangle = \langle u, \frac{1}{\alpha} T u \rangle$

$= \frac{1}{\alpha} \langle u, Tu \rangle$

$= \frac{1}{\alpha} \langle -Tu, u \rangle$ since